

# PROBLEMS IN GROUPS, GEOMETRY, AND THREE-MANIFOLDS

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Dedicated to Daryl Cooper, on the occasion of his 60th birthday.

In May 2015, a conference entitled “Groups, Geometry, and 3-manifolds” was held at the University of California, Berkeley.<sup>1</sup> The organizers asked participants to suggest problems and open questions, related in some way to the subject of the conference. These have been collected here, roughly divided by topic. The name (or names) attached to each question is that of the proposer, though many of the questions have been asked before.

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## 1. CONVEX PROJECTIVE STRUCTURES

1.1. (Agol) If  $M^n$  is a closed manifold with a strictly convex projective structure, is it cubulated? A theorem of Benoist implies that  $\pi_1(M)$  is hyperbolic [4]. This question is open even in the case of closed hyperbolic  $n$ -manifolds.

1.2. (Choi) Suppose a hyperbolic 3-manifold admits a CR-structure (not necessarily spherical). Can the deformation theory of convex real projective structures on  $M$  be understood in terms of the CR-structure? Possibly this is easier for hyperbolic Coxeter 3-orbifolds.

1.3. (Cooper) If  $M$  is a closed hyperbolic 3-manifold is every projective structure on  $M$  convex?

1.4. (Danciger) Let  $N$  be the closed three-manifold obtained by gluing together two copies of figure eight knot complement along their torus boundaries by some homeomorphism. Does  $N$  admit a convex projective structure? (It follows from Ballas–Danciger–Lee [3] that there is a convex projective structure on the *double* of the figure eight knot complement, i.e. if the homeomorphism is chosen to be the identity.)

1.5. (Danciger) More general version: Let  $N$  be a closed three-manifold whose JSJ decomposition contains only hyperbolic pieces. Does  $N$  admit a convex projective structure?

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## 2. GEOMETRIC TRANSITIONS

2.1. (Leitner) Geometric transitions are continuous paths of geometries which abruptly change type in the limit (Cooper-Danciger-Wienhard [13]). The most intuitive example is a sequence of spheres with increasing radius which limit to a plane. It remains to understand all of the transitions between the eight Thurston geometries. More generally, how can one tell when one geometry is a limit of another? One can find a path of transitions to show one geometry limits to another, but it is in general much more difficult to show that one geometry can NOT limit to another. What properties must be satisfied by a limiting geometry?

2.2. (Cooper) Does anyone know of a non-zero polynomial on the tensor product of vector spaces  $U \otimes V \otimes W$  that is invariant under the action of  $SL(U) \times SL(V) \times SL(W)$  when  $\dim(U) = \dim(V) = 4$  and  $\dim(W) = 8$ ? This is related to limits under conjugacy of the diagonal subgroup in  $SL(8, \mathbb{R})$ .

2.3. (Cooper) Suppose  $\beta$  is a non-degenerate bilinear form on a finite dimensional real vector space  $V$  and  $G = \text{Isom}(\beta) \subset GL(V)$ . Which subgroups  $H \subset GL(V)$  are the Hausdorff limits of sequences of conjugates of  $G$ ? This is known for non-degenerate symmetric and skew-symmetric forms.

## 3. SURFACES IN 3-MANIFOLDS

3.1. (Cooper) Are hyperbolic 3-manifolds with large enough injectivity radius Haken [17, Problem 3.58]?

3.2. (Agol) Given a 3-manifold  $M$  whose fundamental group acts on a simplicial tree without global fixed points, consider the covering space associated to the subgroup fixing an edge of the tree. This covering space has a unique 2-dimensional homology class which separates the two ends corresponding to the two ends of the tree minus the edge. Is there a surface in this homology class with minimal Thurston norm that embeds in  $M$ ?

3.3. (Agol) Does every closed hyperbolic 3-manifold contain a closed  $\pi_1$ -injective surface with only double curves of intersection? It seems like this ought to be true for all but finitely many hyperbolic 3-manifolds with volume  $< V$  for any  $V$  by extending results of [14, 18].

3.4. (Agol) Does every closed hyperbolic 3-manifold contain a closed  $\pi_1$ -injective surface which satisfies the 1-line property? This means that in the universal cover, any pair of preimages of the surface intersect in a single line (and the intersection of stabilizers is  $\mathbb{Z}$ ). The surfaces constructed by Kahn and Markovic [16] will likely not have this property, since they tend to overlap on large subsurfaces.

3.5. (Agol) Do cusped finite-volume hyperbolic 3-manifolds have closed quasi-fuchsian surfaces which cubulate except for the cusps? Geometrically, for every pair of points in  $\partial_\infty \mathbb{H}^3$ , some lift of the limit set of a quasi-fuchsian surface should separate this pair. This might be accessible using constructions of Masters-Zhang [21, 22] and Baker-Cooper [2]. One would obtain a cocompact action on a CAT(0) cube complex in which the only point stabilizers are parabolic subgroups [6].

3.6. (Agol) Are finite-volume hyperbolic 3-manifolds virtually semi-fibered?

3.7. (Agol) Which Kleinian groups admit closed quasi-fuchsian surface subgroups? The difficult case is when the group has parabolic subgroups (when there are no parabolics, then this is true if and only if the group is not virtually free).

3.8. (Agol) Let  $M$  be a 3-manifold,  $\phi: \pi_1 M \rightarrow \mathbb{Z}$  dual to the surface  $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ . Let  $N$  be the 3-manifold obtained by cutting  $M$  along  $\Sigma$ . For what  $\alpha$  in  $\text{Hom}(\pi_1 M, SL_2(\mathbb{C}))$  is  $N$  an  $\alpha$ -twisted homology product?

A conjecture of Dunfield–Friedl–Jackson would say that  $N$  is an  $\alpha$ -twisted homology product whenever  $M$  is hyperbolic and  $\alpha$  is discrete faithful.

3.9. (Futer, Manning) Hass’s version of the Simple Loop Conjecture [17, Problem 3.96] states:

**Conjecture 3.1.** *If  $\phi: \Sigma^2 \rightarrow M^3$  is any two-sided immersion of a closed surface of genus  $\geq 1$ , then either  $\phi$  is  $\pi_1$ -injective or compressible. ( $\phi$  is compressible if there is an essential simple closed curve  $\gamma$  on  $\Sigma$  so that  $\phi(\gamma)$  is null-homotopic in  $M$ .)*

As pointed out by Hass, the conjecture is true if and only if it is true for irreducible  $M$ . The conjecture is known for  $M$  a Seifert fibered space (Hass [15]), a graph manifold (Rubinstein-Wang [26]), or a Solv-manifold (recent work of Drew Zemke [29]). It seems not to be known for any hyperbolic manifold, or for any 3-manifold with a nontrivial JSJ with a hyperbolic piece.

Problem: Prove it for some particular hyperbolic manifolds, or find a counterexample.

3.10. (Futer, Schleimer) Is there a practical algorithm to test whether a pair of 3-manifolds are homeomorphic?

3.11. (Walsh) The  $\text{CAT}(0)$  cubed dimension of a group  $G$  is the minimal dimension of a  $\text{CAT}(0)$  cubed space on which  $G$  acts geometrically. Is there a sequence of closed hyperbolic 3-manifold groups, such that their  $\text{CAT}(0)$  cubed dimension tends to infinity? Is there some other sequence of  $\text{CAT}(0)$  groups such that the difference between the  $\text{CAT}(0)$  dimension and the  $\text{CAT}(0)$  cubed dimension tends to infinity? Note, by Bridson [10], there are  $\text{CAT}(0)$  groups such that the difference between the geometric dimension and the  $\text{CAT}(0)$  dimension is arbitrarily large.

#### 4. HYPERBOLIC GROUPS AND GROUPS ACTING ON SPACES

4.1. (Agol) Do closed hyperbolic 3-manifold groups have finite-index subgroups embedding in a word-hyperbolic reflection group?

4.2. (Futer) Gromov asked whether every freely indecomposable hyperbolic group contains the fundamental group of a closed hyperbolic surface. That the answer is yes is sometimes called the *Surface Subgroup Conjecture* for hyperbolic groups. Is this any easier for cubulated hyperbolic groups?

4.3. (Cooper) If  $M$  is a closed 3-manifold when does  $\pi_1 M$  act on the affine building for  $SL(4, \mathbb{R})$  so that the quotient retracts to  $M$ ? An example is  $M = \text{Vol}3$ .

4.4. (Kassel, Mann) Let  $\Gamma$  be a discrete group acting properly discontinuously by affine transformations on  $\mathbb{R}^5$ . Is  $\Gamma$  virtually an extension of a free group by a solvable group?

4.5. (Kassel, Mann) Classify all properly discontinuous affine actions of a given closed surface group on  $\mathbb{R}^6$ .

4.6. (Kassel, Mann) What is the minimal  $n$  such that a given a right-angled Coxeter group admits proper affine action on  $\mathbb{R}^n$ ?

## 5. HYPERBOLIC GEOMETRY/KLEINIAN GROUPS

- 5.1. (Agol) Characterize hyperbolic 3-manifolds with infinitely generated fundamental group. In particular, is there a 3-manifold which is locally hyperbolic with no infinitely-divisible subgroup of the fundamental group (like  $\mathbb{Q}$ ) but not hyperbolic? By locally hyperbolic, we mean any cover with finitely generated fundamental group admits a complete hyperbolic metric, and in particular is tame.
- 5.2. (Agol) Do there exist fibered hyperbolic 3-manifolds which are homology  $S^2 \times S^1$ , and have arbitrarily large injectivity radius? Are there hyperbolic homology spheres of arbitrarily large injectivity radius? See [5], [11]. (cf. [7, 12] for the case of *rational* homology spheres)
- 5.3. (Agol) Characterize the polytopes of the Thurston norms of finite volume hyperbolic 3-manifolds. Thurston completed the rank 2 case, but in higher rank, this is completely open [28].
- 5.4. (Agol) Do hyperbolic 3-manifolds have a finite-sheeted cover homeomorphic to a CAT(0)-cube complex? A good test case might be arithmetic 3-manifolds containing a geodesic surface.
- 5.5. (Agol) Fix a constant  $\mu$  less than the 3-dimensional Margulis constant. Consider the hyperbolic 3-manifolds with volume  $< V$ , and drill out all closed geodesics of length  $< \mu$ . Of the resulting finite collection of 3-manifolds (throwing out repeats), let  $s(V)$  be the fraction of these cusped hyperbolic manifolds which are small (contain no closed incompressible non boundary parallel surface). What is the limiting behavior of  $s(V)$  as  $V \rightarrow \infty$ ? How does it depend on  $\mu$ ?
- 5.6. (Agol) If  $M_1, M_2$  are cusped hyperbolic 3-manifolds, does there exist a cover  $M'_1 \rightarrow M_1$  and a non-zero degree map  $M'_1 \rightarrow M_2$  taking cusps to cusps?
- 5.7. (Agol) The renormalized volume of quasi-fuchsian groups was defined by Schlenker [27]. By Bers' simultaneous uniformization theorem, this gives a function  $\rho : \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathbb{R}$ , where  $\mathcal{T}(S)$  denotes the Teichmüller space of a surface  $S$ . Is  $\rho$  a metric on  $\mathcal{T}(S)$ ?
- 5.8. (Schleimer) Give a rigorous explanation for why SnapPy works so well in practice.
- 5.9. (Cooper) Given  $R > 0$  and an integer  $n \geq 4$  is there  $\epsilon > 0$  and a finite set of hyperbolic  $n$ -simplices with the following property. Consider those closed  $n$ -manifolds which can be constructed by gluing copies of these simplices together by isometries along codimension-1 faces, such the resulting hyperbolic cone  $n$ -manifold has cone angles around each codimension-2 simplex that are in the interval  $(2\pi - \epsilon, 2\pi + \epsilon)$ . Then every such closed  $n$ -manifold admits a hyperbolic metric, and every closed hyperbolic  $n$ -manifold with injectivity radius everywhere bigger than  $R$  is obtained in this way. A set  $P$  with these properties is called a *Thurston's Lego set in  $n$  dimensions*. These exist for  $n \leq 3$ .
- 5.10. (Futer, Schleimer) The Triangulation Conjecture: Every cusped hyperbolic 3-manifold admits a geometric triangulation.
- 5.11. (Futer) Build a combinatorial model for hyperbolic 3-manifolds, with explicit bi-Lipchitz constants.
- 5.12. (Reid) Let  $\Gamma$  be a Kleinian group of finite covolume, and let  $\mathcal{C}(\Gamma)$  be the set of isomorphism classes of the finite groups that are quotients (homomorphic images) of  $\Gamma$ . Does  $\mathcal{C}(\Gamma)$  determine  $\Gamma$  up to isomorphism? (see [24], [9])
- 5.13. (Reid) Let  $\Gamma = F_r$ ,  $r \geq 2$ . Does  $\mathcal{C}(F_r)$  determine  $F_r$  up to isomorphism?
- 5.14. (Reid) Let  $\Gamma$  be a Kleinian group of finite covolume. Does  $\mathcal{C}(\Gamma)$  determine  $\Gamma$  amongst Kleinian groups / fundamental groups of compact 3-manifolds?

5.15. (Gabai, Trnkova) Let  $M$  be a hyperbolic 3-manifold,  $n$  - any positive integer. Does  $M$  admit an ideal triangulation with  $m$  positively oriented tetrahedra, where  $m \geq n$ ?

5.16. (Walsh) If  $G$  is a Gromov hyperbolic group whose boundary is homeomorphic to the limit set of a convex cocompact Kleinian group, is  $G$  virtually a convex cocompact Kleinian group?

Known results: When  $\partial G = S^1$  the answer is yes, and this is due to Tukia, Gabai and Casson–Jungreis. When  $\partial G$  contains no Sierpinski carpet, the answer is again yes, and due to Haissinsky.

If  $\partial G$  does contain a Sierpinski carpet, this is a conjecture of Kapovich–Kleiner, generalizing the Cannon Conjecture, which is the case  $\partial G = S^2$ .

5.17. (Walsh) What subsets of  $S^2$  may occur as limit sets of convex cocompact Kleinian groups?

5.18. (Walsh) For which Kleinian groups does the limit set contain a Sierpinski carpet? For which Kleinian groups does the limit set contain a continuum?

5.19. (Walsh) Characterize limit sets of graph Kleinian groups and iterated graph-Kleinian groups. (A *graph Kleinian group* is a convex co-compact Kleinian group where the double of the convex core is a graph manifold. An *iterated* graph Kleinian group is one whose Bowditch decomposition (see Bowditch, [8]) contains only hanging Fuchsian and graph Kleinian pieces.

## 6. RANDOM QUESTIONS

6.1. (I. Kapovitch) Show that a random walk on the mapping class group gives rise to a pseudo-Anosov element whose invariant foliations have generic trivalent singularities with probability that tends to 1 as the length of the walk tends to infinity.

6.2. (Maher) Start with  $n$  tetrahedra, and glue faces together at random. Note that the links of vertices in this case need not be spheres, but will be (essentially) random triangulated surfaces. Call the resulting space a pseudo-manifold. Investigate properties of pseudo-manifolds.

6.3. (Maher) Rivin [25] obtains many experimental results which seem extremely regular. This may mean there is some sort of additional structure. Investigate.

6.4. (Maher) Consider the orbit of a point  $x$  in Teichmüller space under the action of the mapping class group.

Show that the proportion of orbit points in the ball of radius  $r$  in the Teichmüller metric which are pseudo-Anosov elements whose invariant foliation have generic trivalent singularities tends to 1 as  $r \rightarrow \infty$ .

Show that the proportion of orbit points in the ball of radius  $r$  in the Teichmüller metric which give rise to hyperbolic manifolds when used as Heegaard splitting gluing maps tends to 1 as  $r \rightarrow \infty$ .

## 7. GEOMETRY OF NUMBERS / NUMBER THEORY AND 3-MANIFOLDS

7.1. (Long) Let  $f(x) \in \mathbb{Z}[x]$  be irreducible over  $\mathbb{Q}$  such that all roots are real, and suppose  $f(\alpha) = 0$ . Define  $k = \mathbb{Q}(\alpha)$ , and let  $\mathcal{O}_k$  be the ring of integers. How often is  $\mathcal{O}_k$  a principal ideal domain? Infinitely often?

How often is  $\mathcal{O}_k$  a Euclidean domain, with the standard norm? For quadratic fields this happens only finitely many times.

These questions motivate the consideration of the graphs in the next question.

7.2. (Long) For  $\mathcal{O}_k$  as above we can form a graph  $\Gamma_{\text{unit}}$  whose vertices are elements of  $\mathcal{O}_k$ , so that  $\alpha$  is connected to  $\beta$  by an edge if  $\alpha - \beta$  is a unit. The graph  $\Gamma_{\text{unit}}$  is contained in a graph  $\Gamma_{\text{prime}}$  with the same vertex set, but also containing edges between elements which differ by a prime.

By [19, 3.1],

$$|\text{clique}(\Gamma_{\text{prime}})| \leq |\text{clique}(\Gamma_{\text{unit}})| \cdot \min\{|\mathcal{O}_k/I| \mid I \text{ prime}\}$$

Is this an equality?

7.3. (Manning) If  $k$  is a number field which is not totally real, is there a hyperbolic 3-manifold with trace field  $k$ ? This is an old question of Neumann–Reid. See [20, 23] for further information.

7.4. (Agol) Can there be a degenerate Kleinian group which is not the fiber of a fibration and has algebraic trace field [1]?

7.5. (Schleimer) More specifically, is there a singly degenerate Kleinian group where all of the matrix entries of the group elements lie in a fixed number field?

7.6. (McMullen) Let  $M$  be a finite volume hyperbolic 3-manifold. Suppose  $M$  contains infinitely many immersed totally geodesic surfaces. Then is  $M$  arithmetic?

## 8. HEEGARD SPLITTINGS, MULTISECTIONS AND KNOTS

8.1. (Agol) Do Haken hyperbolic 3-manifolds have strongly irreducible Heegaard splittings?

8.2. (Dunfield) If you have a hyperbolic 3-manifold with torus boundary, does its profinite completion determine whether or not it's a knot complement?

8.3. (Futer, Schleimer) Can the unknot be detected in polynomial time? Similarly for  $S^3$ .

8.4. (Schleimer) Is there an algorithm to detect whether a Heegaard splitting is reducible (and if so, find a reducing curve)?

8.5. (Schleimer) Is there a classification of strongly irreducible Heegaard splittings of a given 3-manifold? Update: See <http://arxiv.org/abs/1509.05945> for recent progress.

8.6. (Schleimer) Is there a fast algorithm to compute distances in the curve complex? How about distances between quasiconvex sets, such as handlebody sets?

8.7. (Tillmann) Theorem: (Gay, Kirby)

- Every closed, orientable smooth 4-manifold has a trisection.
- Any two trisections have a common stabilization.

Do higher dimensional smooth manifolds always have multisections? Do What is the right generalization of “uniqueness up to stabilization” for multisections of smooth  $n$ -manifolds where  $n \geq 5$ ?

8.8. (Taylor) Given a 2-component link  $L$  in the 3-sphere, we can attach a band joining the components in such a way so as to end up with a knot  $K$  (which depends on both the original link and the choice of band). Say that the band is “complicated” if one of the following holds:

- The link  $L$  is a split link and the core of the band cannot be isotoped to intersect a splitting sphere in fewer than 3 points
- The core of the band cannot be isotoped to be disjoint from any minimal genus Seifert surface for the link.

In a recent paper I showed that a knot  $K$  arising by attaching a complicated band satisfies the Cabling Conjecture. What is an example of a hyperbolic knot which cannot be created by attaching a complicated band to a 2-component link?

8.9. (Tillmann) Can every 2-link in  $\mathbb{S}^4$  be put in bridge position with respect to the standard trisection of  $\mathbb{S}^4$ ? (Update: The paper <http://arxiv.org/abs/1507.08370> gives an answer to this question.)

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